Mathematics: analysis and approaches
Higher level
Paper 1

Topic: Derivatives

Candidate name

75 minutes

Instructions to candidates

- Write your name in the boxes above.
- Do not open this examination paper until instructed to do so.
- You are not permitted access to any calculator for this paper.
- Answer all questions.
- Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.
- Unless otherwise stated in the question, all numerical answers should be given exactly or correct to three significant figures.
- The maximum mark for this examination paper is **[53 marks]**.

8 pages

Answer **all** questions. Answers must be written within the answer boxes provided. Working may be continued below the lines, if necessary.

1. [Maximum mark: 5]

Using the first principle, show that the derivative of $(x + 1)^3$ is $3(x + 1)^2$.

$$\lim_{h \to 0} \frac{(x+h+1)^3 - (x+1)^3}{h} \quad M1$$

$$= \lim \frac{\cancel{(x+1)^3} + 3h(x+1)^2 + 3h^2(x+1) + h^3 - \cancel{(x+1)^3}}{h} \quad M1 \quad A1$$

$$= \lim 3(x+1)^2 + 3h(x+1) + h^2 \quad M1$$

$$= 3(x+1)^2 \quad A1$$

2. [Maximum mark: 6]

Find the limit: $\lim_{x \to 0} \frac{\sec^4 x - \cos^2 x}{x^4 - x^2}$.

$= \lim_{x \to 0} \frac{4\sec^3 x \tan x + 2\cos x \sin x}{4x^3 - 2x}$ M1 A1

$= \lim_{x \to 0} 4\sec^3 x \cdot \frac{\cancel{\tan x}}{\cancel{x}(4x^2 - 2)} + \lim_{x \to 0} \frac{\cancel{\sin 2x}}{\cancel{2x}(4x^2 - 2)} \cdot 2$ M1 A1

$= \lim_{x \to 0} \frac{4\sec^3 x + 2}{4x^2 - 2}$ A1

$= \frac{4+2}{-2} = -3$ A1

3. [Maximum mark: 8]

Consider the curve with the equation $x^3 + 4y^3 = xy$.

(a) Find an expression for $\frac{dy}{dx}$. [3]

(b) Find the coordinates of all points where the tangent to the curve is vertical. [5]

(a) $3x^2 + 12y^2 y' = y + xy'$ M1 differentiate both side, A1 product rule

$(12y^2 - x)y' = y - 3x^2$

$y' = \dfrac{y - 3x^2}{12y^2 - x}$ A1

(b) M1

$\begin{cases} 12y^2 - x = 0 \Rightarrow x = 12y^2 \quad ① \\ x^3 + 4y^3 = xy \quad ② \end{cases}$

Sub ① into ②: M1

$12^3 \cdot y^6 + 4y^3 = 12y^3$

$12^3 y^6 - 8y^3 = 0$

$y^3(12^3 y^3 - 8) = 0$

$\Rightarrow y = 0$ A1 or $12^3 y^3 = 8 \Leftrightarrow y = \frac{2}{12}$

$= \frac{1}{6}$ A1

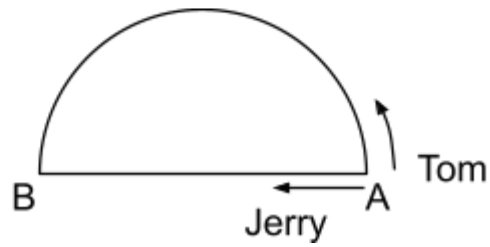
Sub back:

$x = 0$ or $x = 12 \cdot \frac{1}{6^3} = \frac{2}{36} = \frac{1}{18}$

$\Rightarrow (0, 0)$ or $\left(\frac{1}{6}, \frac{1}{18}\right)$ A1

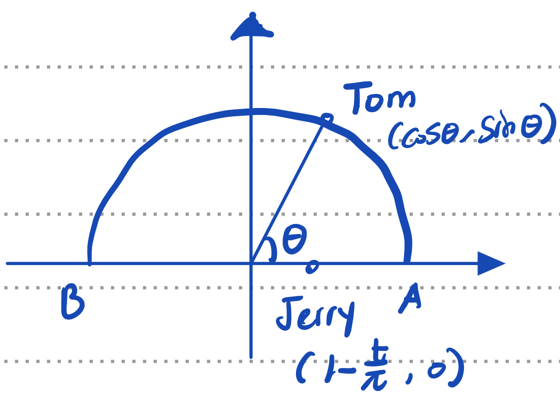
4. [Maximum mark: 8]

Tom and Jerry travels from point A to point B along two different paths: Tom travels along the semi-circle with radius of 1 km at the speed of 1 km/h, and Jerry travels along the diameter AB at the speed of $\frac{1}{\pi}$ km/h.



Find the rate of change of the distance between Tom and Jerry at the time when Jerry travels half way through.

Construct a coordinate system:



at the time t hours,

Tom travelled $t \times 1 = t$ km

$\Rightarrow \theta l = t$ M1

$\Rightarrow \theta = t$ A1

$\Rightarrow$ coordinate of Tom:

$(\cos t, \sin t)$ A1

$\Rightarrow$ distance $x = \sqrt{(\cos t - (1-\frac{t}{\pi}))^2 + \sin^2 t}$

$= \sqrt{\cos t^2 - 2\cos t(1-\frac{t}{\pi}) + (1-\frac{t}{\pi})^2 + \sin^2 t}$

$= \sqrt{1 - 2\cos t + 2\cos t \cdot \frac{t}{\pi} + (1-\frac{t}{\pi})^2}$ M1

$x^2 = 1 - 2\cos t + 2\cos t \cdot \frac{t}{\pi} + (1-\frac{t}{\pi})^2$

$2x \cdot \frac{dx}{dt} = 2\sin t + (-2\sin t \cdot \frac{t}{\pi} + \frac{2}{\pi}\cos t) - \frac{2}{\pi}(1-\frac{t}{\pi})$

A1

When Jerry is mid way through, $t = \frac{1}{\pi} = \pi$ A1

$\cancel{2}x(\pi) \cdot \frac{dx}{dt} = \frac{\cancel{2}}{\pi} - \frac{\cancel{2}}{\pi}\left(1 - \frac{\pi}{\pi}\right)$ M1

$\frac{dx}{dt} = \frac{1}{\pi}$ A1

5. [Maximum mark: 7]

Consider the function $f(x) = \sqrt{x^2 ln(x) + 4 - x^2}$, where $x \in \mathbb{R}, x > 0$.

(a) Show that the distance , l, between the origin and any point on the graph is given by $l = \sqrt{x^2 ln(x) + 4}$. [1]

(b) Hence, find the x-coordinate of the point on the graph of f which is the closest to the origin. [6]

(a) $d = \sqrt{f^2(x) + x^2}$

$= \sqrt{x^2 \ln x + 4}$ M1

(b) $d' = \dfrac{2x\ln x + x}{2\sqrt{x^2\ln x + 4}}$ M1 chain rule, M1 product rule, A1

Set $d' = 0$: M1

$2x\ln x + x = 0$

$2\ln x + 1 = 0$ M1

$\ln x = -\frac{1}{2}$

$x = e^{-\frac{1}{2}}$ A1

Answer **all** questions in the answer booklet provided. Please start each question on a new page.

6. [Maximum mark: 19]

Consider the curve C defined by the equation $e^{x+y} - 1 = x^2 + y^2$.

(a) Show that $\frac{dy}{dx} = \frac{2x - e^{x+y}}{e^{x+y} - 2y}$. [5]

(b) Show that there is no point on the graph where the tangent is horizontal. [7]

(c) Show that the graph of C is symmetric about the line $y = x$. [3]

(d) Find the coordinates of the point on the curve C where the tangent has a gradient of -1. [4]

(a) $e^{x+y}(1+y') = 2x + 2yy'$

M1: differentiate both side
M1: chain rule
A1

$(e^{x+y} - 2y)\,y' = 2x - e^{x+y}$ A1

$y' = \frac{2x - e^{x+y}}{e^{x+y} - 2y}$ A1

(b) Assume, by contradiction, there's point with horizontal tangent. M1

$\Rightarrow \begin{cases} 2x - e^{x+y} = 0 \Rightarrow e^{x+y} = 2x & ① \\ e^{x+y} - 1 = x^2 + y^2 & ② \end{cases}$ M1

Sub ① to ② M1

$2x - 1 = x^2 + y^2$ A1

$0 = x^2 - 2x + 1 + y^2 = (x-1)^2 + y^2$ M1

$\Rightarrow \quad x-1=0 \quad \& \quad y=0$

$\Rightarrow \quad (x,y) = (1,0)$ A1

but $(1,0)$ is not on the curve as

$e^0+1 \neq 0+0$ (contradiction) A1

(c) swap x & y we obtain: M1

$$e^{y+x}-1 = y^2+x^2$$ A1

which is the same curve as the original. A1

(d) $\frac{dy}{dx} = -1$ M1

$\Rightarrow \quad x=y$ A1

$\Rightarrow \quad e^{2x}-1 = 2x^2$

Note $x=0$ is the solution A1

$\Rightarrow \quad x=y=0$ A1